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SENSITIVITY ANALYSIS FOR PROBLEMS OF DYNAMIC STABILITY

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Abstract—In mechanics, as well as in physics, the most general and important thing is to study the dependence of the characteristics of a physical process on problem parameters. Problems of dynamic stability for non-conservative systems involve determination of eigenvalues and eigenvectors. For these problems it is shown in general how the different sensitivity analyses can be performed without any new eigenvalue analyses. The main question relates to the change in flutter load as a function of change in stiffness, mass, boundary conditions, or load distribution. Discretized as well as non-discretized examples are presented in details.

INTRODUCTION

Problems of stability for non-conservative systems are connected with the question of stability of vibrations, i.e. dynamic stability. There are several classic works on this subject: Bolotin[1], Ziegler[2] and Leipholz[3].

The condition of stability (boundaries between stable and unstable domains) naturally depends on the parameters of the problem, i.e. on the design, on the boundary conditions, the load distribution, etc., and it is therefore important to obtain qualitative and also quantitative information about the dependence of stability on the discrete and distributed parameters of the problem. This is what we refer to as sensitivity analysis. The intention of this paper is to present detailed analyses of the sensitivity of dynamic stability in the sense of derivatives with respect to discrete parameters, and gradient functions for the case when the independent parameter is a function.

Dealing with non-conservative problems we have to be aware that instability may occur either dynamically (flutter) or statically (divergence). Also we have to note that the classic extremum principles like the Rayleigh principle are not valid for non-conservative problems. However, recent analyses have shown that the introduction of the adjoint problem admits a stationarity principle. This theory has been described by Leipholz[4], and has also been dealt with by Prasad and Herrmann[5, 6]. In relation to the sensitivity analysis, this introduction of the adjoint problem is of major importance.

It is interesting to note that the sensitivity analysis with respect to an increment of the load parameter clarifies the condition of instability, and thus makes possible a more rigorous definition of terms like critical load, flutter load and divergence load. For this sensitivity analysis as well as all the other sensitivity analyses performed, it turns out that the solution to the main and the adjoint problem provides all the necessary information for evaluating the sensitivities. Thus, we may increase our level of information without too much additional effort.

One major application of sensitivity analysis is within optimal structural design, because sensitivities provide the necessary information for advantageous redesign. Also, it should be noted that optimality conditions/criteria are given directly by sensitivities. In the proceedings[7] (mostly dealing with conservative problems) the editors Haug & Cea treat sensitivity analysis as the "cornerstone" of any approach to optimal design, and the present authors agree with this point of view.

Optimal design for non-conservative problems has been studied by Ashley and

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McIntosh[8, 9], Vepa[10], Plaut and Weisshaar[11, 12], Claudon[13–15], Odeh and Tadjbakhsh[16], Hanaoka and Washizu[17]. Unfortunately, many papers are confusing with respect to the expressions for the gradient functions. Although these gradient functions include much information, the reader is seldom shown how these functions look for a specific design. Gradient functions are displayed in papers by Seyranian[18] for non-conservative problems and Pedersen[19] for conservative problems.

The intention of the present paper is to derive more general expressions for different derivatives and gradient functions, and then to discuss the detailed results for many specific examples.

2. SENSITIVITY ANALYSIS IN MATRIX FORMULATION

Let us consider linear non-conservative systems with the finite degree of freedom n, focusing on the phenomenon of dynamic stability. Thus we shall present sensitivity analysis in matrix formulation, which has the advantage that boundary conditions are an integrated part. Furthermore, the formulation relates directly to practical methods of analysis like the finite element method. For continuous description with operator formulation, see Seyranian[18]. We shall assume that the system is characterized by the parameters t_i, including the different design variables

$$\{t_1, t_2, \ldots, t_m\}, \tag{2.1}$$

and we shall frequently omit the index of a specific parameter t.

Separating the time dimension τ by the exponential function

$$e^{\lambda\tau} = e^{(\alpha+i\omega)\tau} = e^{\alpha\tau}\, e^{i\omega\tau} = e^{\alpha\tau}(\cos\omega\tau + i\sin\omega\tau), \tag{2.2}$$

we get the homogeneous matrix equation

$$[L]\{\Phi\} = \{0\} \tag{2.3}$$

where the system matrix $[L]$ depends on the real load parameter P, on the complex eigenvalue $\lambda = \alpha + i\omega$, and on the real parameters t_i. The complex eigenvector $\{\Phi\}$ contains the generalized coordinates, say displacements. We may write the matrix $[L]$ as a linear function of real matrices, specifying the dependence on λ explicitly by

$$[L] = [S] + [Q] + \lambda^2[M] + \lambda[C], \tag{2.4}$$

where the symmetric stiffness matrix $[S]$ and the symmetric mass matrix $[M]$ depend on t_i, but not on P. The nonsymmetric load matrix $[Q]$ and the damping matrix $[C]$ depend on P as well as on t_i.

The dynamic stability of the system is determined quantitatively by the eigenvalues $\lambda_r = \alpha_r + i\omega_r$ for $r = 1, 2, \ldots, n$. The eigenvalue with the maximum real part is the important one, and for this the index is omitted. From the time function (2.2) we have

$$\begin{gathered} \alpha \geq \alpha_r \text{ for } r = 1, 2, \ldots, n \\ \text{Stable motion for } \alpha < 0 \\ \text{Critical motion for } \alpha = 0 \\ \text{Flutter instability for } \alpha > 0 \text{ and } \omega \neq 0 \\ \text{Divergence instability for } \alpha > 0 \text{ and } \omega = 0 \end{gathered} \tag{2.5}$$

The critical case is clarified by determining the derivative of α, i.e. by a sensitivity analysis. A more precise definition of a critical load $P = P_{cr}$ is therefore

$$\alpha = 0 \text{ and } \partial\alpha/\partial P > 0 \text{ at } P = P_{cr} \tag{2.6}$$

This is a sufficient condition for instability of the system, when P becomes greater than P_{cr}.

Sensitivities of the critical load

In addition to the physical problem (2.3) we define an adjoint problem by

$$[L]^T\{\Psi\} = \{0\} \text{ or } \{\Psi\}^T[L] = \{0\}^T \tag{2.7}$$

where the complex eigenvector $\{\Psi\}$ contains the adjoint generalized coordinates. The eigenvalues of (2.3) and (2.7) are identical.

Let us take a variation δt of the system parameter t. Then the quantities λ, P, $\{\Phi\}$ undergo the increments $\delta\lambda$, δP, $\{\delta\Phi\}$, and using the symbol of "," for partial differentiation, eqn (2.3) gives

$$[L]_{,\lambda}\{\Phi\}\delta\lambda + [L]_{,P}\{\Phi\}\delta P + [L]_{,t}\{\Phi\}\delta t + [L]\{\delta\Phi\} = \{0\}. \tag{2.8}$$

We multiply this equation from the left by $\{\Psi\}^T$, and by the definition (2.7) get $\{\Psi\}^T[L]\{\delta\Phi\} = 0$, whereby eqn (2.8) gives

$$(\{\Psi\}^T[L]_{,\lambda}\{\Phi\})\delta\lambda + (\{\Psi\}^T[L]_{,P}\{\Phi\})\delta P + (\{\Psi\}^T[L]_{,t}\{\Phi\})\delta t = 0. \tag{2.9}$$

Especially at a critical load $P = P_{cr}$ we have by condition (2.6) $\lambda = i\omega_{cr}$ and $\delta P = \delta P_{cr}$, $\delta\lambda = i\delta\omega_{cr}$, whereby eqn (2.9) gives

$$(\{\Psi\}^T[L]_{,P}\{\Phi\})\delta P_{cr} + (\{\Psi\}^T[L]_{,t}\{\Phi\})\delta t = -i(\{\Psi\}^T[L]_{,\lambda}\{\Phi\})\delta\omega_{cr}. \tag{2.10}$$

For divergence instability, where $\lambda_{cr} = i\omega_{cr} = 0$ and $\delta\lambda_{cr} = i\delta\omega_{cr} = 0$, eqn (2.10) gives the derivative $\partial P_{cr}/\delta t$ directly, but for flutter instability we have to take the real parts, whereby we get the result

$$\frac{\partial P_{cr}}{\partial t} = -Re\left(\frac{\{\Psi\}^T[L]_{,t}\{\Phi\}}{\{\Psi\}^T[L]_{,\lambda}\{\Phi\}}\right)\Big/ Re\left(\frac{\{\Psi\}^T[L]_{,P}\{\Phi\}}{\{\Psi\}^T[L]_{,\lambda}\{\Phi\}}\right) \tag{2.11}$$

which is also valid for divergence instability.

Note, that both the derivative $\partial P_{cr}/\partial t$, and the derivatives to follow are determined by the system matrix $[L]$, the eigenvector $\{\Phi\}$ and the adjoint eigenvector $\{\Psi\}$ without involving further (expensive computer) calculations.

Sensitivities of the eigenvalue

The sensitivity of the critical frequency $\delta\omega_{cr}$ also contain important information, and as δP_{cr} is a real quantity, from eqn (2.10) we find this to be

$$\frac{\partial\omega_{cr}}{\partial t} = -Im\left(\frac{\{\Psi\}^T[L]_{,t}\{\Phi\}}{\{\Psi\}^T[L]_{,P}\{\Phi\}}\right)\Big/ Re\left(\frac{\{\Psi\}^T[L]_{,\lambda}\{\Phi\}}{\{\Psi\}^T[L]_{,P}\{\Phi\}}\right) \tag{2.12}$$

If the load is kept unchanged, we get from eqn (2.9) with $\underline{\delta P = 0}$

$$\frac{\partial\lambda}{\partial t} = \frac{\partial\alpha}{\partial t} + i\frac{\partial\omega}{\partial t} = \frac{-\{\Psi\}^T[L]_{,t}\{\Phi\}}{\{\Psi\}^T[L]_{,\lambda}\{\Phi\}}. \tag{2.13}$$

If on the other hand, the system parameters are kept unchanged, we get from eqn (2.9) with $\underline{\delta t = 0}$

$$\frac{\partial\lambda}{\partial P} = \frac{\partial\alpha}{\partial P} + i\frac{\partial\omega}{\partial P} = \frac{-\{\Psi\}^T[L]_{,P}\{\Phi\}}{\{\Psi\}^T[L]_{,\lambda}\{\Phi\}}. \tag{2.14}$$

The instability criterion (2.6) may therefore be written

$$\alpha = 0 \quad \text{and} \quad Re\left(\frac{\{\Psi\}^T[L]_{,P}\{\Phi\}}{\{\Psi\}^T[L]_{,\lambda}\{\Phi\}}\right) < 0. \tag{2.15}$$

The true but more complicated formulas corresponding to (2.11) and (2.12) were first obtained by Bun'kov[20]. The formulas also agree with the ones in the paper by Claudon and Sunakawa[15] (except for a sign in eqn (8) of [15]). The reader should be aware of the fact that many earlier papers contain errors in their corresponding expressions.

3. SYSTEMS WITHOUT DAMPING

For systems without damping the results of the previous section can be considerably simplified. The system matrix $[L]$ reduces to

$$[L] = [S] + [Q] + \lambda^2[M], \tag{3.1}$$

and the solutions to (2.3) are either real λ^2 with corresponding real $\{\Phi\}$ ($\lambda^2 > 0$ giving divergence instability and $\lambda^2 < 0$ stable harmonic vibrations), or complex conjugate solutions with corresponding complex conjugate eigenvectors $\{\Phi\}$.

Let us write the "mutual energies" corresponding to two different solutions i and j of (2.3) and (2.7), respectively

$$\begin{aligned} &\{\Psi\}_j^T[S]\{\Phi\}_i + \{\Psi\}_j^T[Q]\{\Phi\}_i + \lambda_i^2\{\Psi\}_j^T[M]\{\Phi\}_i = 0, \\ &\{\Psi\}_j^T[S]\{\Phi\}_i + \{\Psi\}_j^T[Q]\{\Phi\}_i + \lambda_j^2\{\Psi\}_j^T[M]\{\Phi\}_i = 0. \end{aligned} \tag{3.2}$$

Then, by subtraction, we get

$$(\lambda_i^2 - \lambda_j^2)(\{\Psi\}_j^T[M]\{\Phi\}_i) = 0 \tag{3.3}$$

which, for $\lambda_i^2 \neq \lambda_j^2$, gives the biorthogonality condition

$$\{\Psi\}_j^T[M]\{\Phi\}_i = 0 \quad \text{for} \quad i \neq j \tag{3.4}$$

Then by arguments of continuity, at a flutter point, where two eigenvalues become equal, we get the "flutter condition"

$$\{\Psi\}_F^T[M]\{\Phi\}_F = 0 \tag{3.5}$$

and therefore by (3.1)

$$\{\Psi\}_F^T[L]_{,\lambda}\{\Phi\}_F = 2\lambda_F\{\Psi\}_F^T[M]\{\Phi\}_F = 0. \tag{3.6}$$

For divergence instability eqn (3.6) is also valid because $\lambda_D = 0$. Therefore, we get the simplified expression for systems without damping from (2.9)

$$\frac{\partial P_{cr}}{\partial t} = \frac{-\{\Psi\}^T[L]_{,t}\{\Phi\}}{\{\Psi\}^T[L]_{,P}\{\Phi\}}, \tag{3.7}$$

in agreement with Hanaoka and Washizu[17].

In systems modelled by the finite element method, the system matrix $[L]$ may often be written as

$$[L] = t^n[\hat{S}] + P[\hat{Q}] + \lambda^2 t^m[\hat{M}] + [\hat{L}], \tag{3.8}$$

where the matrices $[\hat{S}]$, $[\hat{Q}]$, $[\hat{M}]$, $[\hat{L}]$ are independent of t and P. Thus, the derivative (3.7) will be

$$\frac{\partial P_{cr}}{\partial t}=\frac{(nt^n\{\Psi\}^T[\hat{S}]\{\Phi\}+mt^m\lambda^2\{\Psi\}^T[\hat{M}]\{\Phi\})/t}{-\{\Psi\}^T[\hat{Q}]\{\Phi\}}. \tag{3.9}$$

Each term of eqn (3.9) may be interpreted as a specific mutual energy. Formula (2.13) for this case with $\lambda^2 = -\omega^2$ simplifies to

$$\frac{\partial \omega}{\partial t}=\frac{(nt^n\{\Psi\}^T[\hat{S}]\{\Phi\}-mt^m\omega^2\{\Psi\}^T[\hat{M}]\{\Phi\})/t}{2\omega\{\Psi\}^T[M]\{\Phi\}}. \tag{3.10}$$

This formula is used in [19] for selfadjoint problems where $\{\Psi\}=\{\Phi\}$. Analogously, eqn (2.14) reduces to

$$\frac{\partial \omega}{\partial P}=\frac{\{\Psi\}^T[\hat{Q}]\{\Phi\}}{2\omega\{\Psi\}^T[M]\{\Phi\}}. \tag{3.11}$$

4. A NON-DISCRETIZED PROBLEM

Figure 1 shows a follower force problem, which is an extension of the classic Beck problem: extended to include a linear elastic support, a concentrated mass in addition to the distributed mass, and a partial follower force. The stability analysis for this problem is given in Ref.[21], and here we shall give the formulation for a non-uniform column. For this problem the separated differential equation in space is

$$(su'')''+pu''+\lambda^2 mu=0 \tag{4.1}$$

and the boundary conditions are

$$\begin{gathered} u(0)=u'(0)=su''(1)=0 \\ (su'')'(1)+(1-\eta)pu'(1)-(\mu\lambda^2+\kappa)u(1)=0 \end{gathered} \tag{4.2}$$

as shown in detail in[21]. For a uniform column the non-dimensional stiffness and mass are $s\equiv 1$ and $m\equiv 1$. If the non-dimensional follower force p is known, (4.1)–(4.2) will constitute an eigenvalue problem for the "frequency" quantity $\lambda^2=(\alpha+i\omega)^2$ which is based on the non-dimensional time separation $e^{(\alpha+i\omega)\tau}$. Note that p as well as λ^2 is involved in the boundary

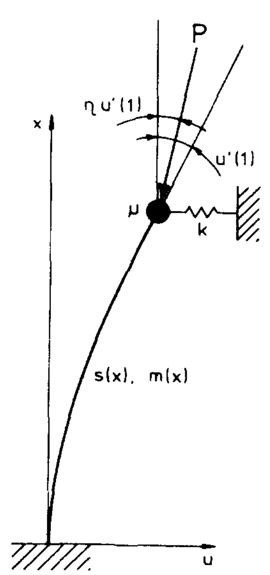

Fig. 1. Extended Beck column.

condition, which also reflects the elastic support stiffness κ, the concentrated mass μ and the follower angle η.

For the uniform column it is possible to solve this eigenvalue problem without discretization, and the eigenfunction may be written

$$u(x, p, \lambda) = y + fz \tag{4.3}$$

with

$$\begin{aligned} y(x, p, \lambda) &= \cosh(ax) - \cos(bx) \\ z(x, p, \lambda) &= a \sin(bx) - b \sinh(ax) \\ f(p, \lambda) &= \frac{a^2 \cosh(a) + b^2 \cos(b)}{ab(a \sinh(a) + b \sin(b))} \\ \begin{matrix} a \\ b \end{matrix} &= \sqrt{\left(\mp\frac{p}{2} + \sqrt{\left(\frac{p^2}{4} - \lambda^2\right)}\right)}. \end{aligned} \tag{4.4}$$

The problem, which is adjoint to (4.1)–(4.2), is described by the differential equation (equal to (4.1))

$$(sv'')'' + pv'' + \lambda^2 mv = 0 \tag{4.5}$$

and the boundary conditions

$$\begin{aligned} v(0) = v'(0) &= 0 \\ sv''(1) + \eta p v(1) &= 0 \\ (sv'')'(1) + pv'(1) - (\mu\lambda^2 + \kappa)v(1) &= 0 \end{aligned} \tag{4.6}$$

and the eigenfunction for the case of a uniform column is

$$v(x, p, \lambda) = y + gz \tag{4.7}$$

with

$$\begin{aligned} g &= \frac{(a^2 - \eta p)\cosh(a) + (b^2 - \eta p)\cos(b)}{b(a^2 + \eta p)\sinh(a) + a(b^2 - \eta p)\sin(b)} \\ &= \frac{a(a^2 + p)\sinh(a) - b(b^2 - p)\sin(b) - (\mu\lambda^2 + \kappa)(\cosh(a) - \cos(b))}{ab((a^2 + p)\cosh(a) + (b^2 - p)\cos(b)) - (\mu\lambda^2 + \kappa)(b \sinh(a) - a \sin(b))}. \end{aligned} \tag{4.8}$$

Note that, for the conservative problem ($\eta = 0$), we have equality between (4.2) and (4.6), and $g = f$ in agreement with $v \equiv u$.

The solution to this stability problem $\lambda = \lambda(p, \eta, \mu, \kappa)$ is given in[21]. We shall concentrate on the flutter point solutions $\lambda_F, p_F = \lambda_F, p_F(\eta, \mu, \kappa)$ listed in Table 1, and show how sensitivities are obtained without further eigenvalue solutions.

First we derive the "flutter condition" for problem (4.1)–(4.2). We multiply by v and integrate to obtain the "mutual potential and kinetic energies"

$$\int (sv''u'' - pv'u')\,dx + (\eta pvu' + (\mu\lambda_u^2 + \kappa)vu)_{x=1} + \int \lambda_u^2 mvu\,dx = 0, \tag{4.9}$$

where on integration by parts the boundary conditions (4.2) are used. The eigenvalue is given the index u to indicate the relation to the eigenfunction u. Analogously, from the adjoint

Table 1. Flutter load p_F and flutter frequency ω_F for the extended Beck problem, from [21]

	$\mu=\kappa=0$			$\eta=1$, $\kappa=0$				$\eta=1$, $\mu=0$			
	$\eta=0.5$	$\eta=1$	$\eta=1.5$	$\mu=0$	$\mu=0.5$	$\mu=1$	$\mu=2$	$\kappa=0$	$\kappa=10$	$\kappa=20$	$\kappa=30$
p_F	16.1	20.05	30.6	20.05	16.1	16.2	16.6	20.05	24.5	30	36
ω_F	7.1	11	12	11	7.3	6.3	5.5	11	10.9	10	8.3

problem (4.5)–(4.6), we get

$$\int (su''v''-pu'v')\,dx+(\eta pu'v+(\mu\lambda_v^2+\kappa)uv)_{x=1}+\int\lambda_v^2 muv\,dx=0. \tag{4.10}$$

Subtracting (4.10) from (4.9) yields

$$(\lambda_u^2-\lambda_v^2)\left(\int muv\,dx+\mu(uv)_{x=1}\right)=0 \tag{4.11}$$

which for $\lambda_u\neq\lambda_v$ expresses the biorthogonality condition. At a flutter point the condition of a double eigenvalue, by arguments of continuity, gives the flutter condition

$$\int mu_Fv_F\,dx+\mu(u_Fv_F)_{x=1}=0. \tag{4.12}$$

Here we have a condition which includes a boundary term.

That the problems (4.1)–(4.2) and (4.5)–(4.6) are adjoint is seen directly from (4.9)–(4.10) by putting $\lambda_u=\lambda_v$.

Stationarity of the total mutual energies, as discussed generally in section two, means that we may obtain the variation of eqn (4.9) without varying the displacements u and v. If all other quantities are varied we obtain

$$\int \delta sv''u''\,dx-\delta p\int v'u'\,dx+(\delta\eta pvu'+\delta p\eta vu'+\delta\mu\lambda^2vu+\delta\lambda\mu 2\lambda vu+\delta\kappa vu)_{x=1}$$
$$+\delta\lambda 2\lambda\int mvu\,dx+\lambda^2\int\delta mvu\,dx=0 \tag{4.13}$$

which, at a flutter point where $\lambda_F^2=-\omega_F^2\,(\alpha_F=0)$ and where condition (4.12) is valid, simplifies to

$$\int\delta sv_F''u_F''\,dx-\omega_F^2\int\delta mv_Fu_F\,dx+(\delta\eta p_Fv_Fu_F'-\delta\mu\omega_F^2v_Fu_F+\delta\kappa v_Fu_F)_{x=1} \tag{4.14}$$
$$=\delta p_F\left(\int v_F'u_F'\,dx-\eta(v_Fu_F')_{x=1}\right)$$

The derivatives of the flutter load p_F with respect to changes of boundary parameters η, μ and κ are therefore

$$\partial p_F/\partial\eta=p_F(v_Fu_F')_{x=1}\Big/\left(\int v_F'u_F'\,dx-\eta(v_Fu_F')_{x=1}\right)$$
$$\partial p_F/\partial\kappa=(v_Fu_F)_{x=1}\Big/\left(\int v_F'u_F'\,dx-\eta(v_Fu_F')_{x=1}\right) \tag{4.15}$$
$$\partial p_F/\partial\mu=-\omega_F^2\,\partial p_F/\partial\kappa$$

and for the design changes δs, δm we get

$$\delta p_F = \frac{\int \delta s v_F'' u_F'' \mathrm{d}x - \omega_F^2 \int \delta m v_F u_F \, \mathrm{d}x}{\int v_F' u_F' \mathrm{d}x - \eta(v_F u_F')_{x=1}} \tag{4.16}$$

which, for the case of $s = m^2$, by defining the gradient function

$$g(x) = \frac{2mv_F'' u_F'' - \omega_F^2 v_F u_F}{\int v_F' u_F' \mathrm{d}x - \eta(v_F u_F')_{x=1}} \tag{4.17}$$

we may write

$$\delta p_F = \int g \delta m \, \mathrm{d}x. \tag{4.18}$$

Table 2 shows some results of the derivatives (4.15), and Figs. 2–4 shows gradient functions calculated by (4.17).

It may be observed that the influence of an end-spring or an end-mass is evaluated without analysis of a column including these effects. In total, if we have increments $\delta\eta$, $\delta\kappa$, $\delta\mu$, $\delta m(x)$ we can obtain the total increment of p_F by

$$\delta p_F = \frac{\partial p_F}{\partial \eta} \delta\eta + \frac{\partial p_F}{\partial \kappa} \delta\kappa + \frac{\partial p_F}{\partial \mu} \delta\mu + \int_0^1 g \delta m \, \mathrm{d}x. \tag{4.19}$$

In several earlier papers the term $- \omega_F^2 v_F u_F$ of the gradient function (4.17) is missing, which has encouraged us to show specifically the influence of this term. Furthermore, according to eqn (4.16), this term gives the gradient function for non-structural masses ($\delta s = 0$, $\delta m \neq 0$).

Results with a partial follower force (parameter η of Fig. 1) are shown in Fig. 2, and we see a very strong influence of η especially in relation to the term $- \omega_F^2 v_F u_F$. Note that in domains of a negative gradient ($g(x) < 0$), the flutter load p_F is increased by removing material ($\delta m(x) < 0$).

The influence of an end-mass (parameter μ in Fig. 1) is shown in Fig. 3. The influence is rather weak and the main thing to be noted is that the importance of the term ($- \omega_F^2 v_F u_F$) has diminished.

Finally, we have studied the influence of an end-spring (parameter κ in Fig. 1) and give the results in Fig. 4. This parameter introduces a second domain of high positive $g(x)$, and thus modifies the gradient function more generally.

5. SOME DISCRETIZED PROBLEMS

Only few problems like the one in section four can be solved directly without discretization. When the column is non-uniform or the load is distributed, we have to perform the stability analysis by the finite difference method, the finite element method, the Ritz method, the Bubnov–Galerkin method or another weighted residual method. To solve the problems corresponding to Fig. 5 we shall here use the last-named method, but in reality the general aspects of the results do not depend on the discretization method chosen.

Table 2. Derivatives of the flutter load with respect to boundary parameters

	$\mu = \kappa = 0$			$\eta = 1$, $\kappa = 0$				$\eta = 1$, $\mu = 0$			
	$\eta = 0.5$	$\eta = 1$	$\eta = 1.5$	$\mu = 0$	$\mu = 0.5$	$\mu = 1.0$	$\mu = 2.0$	$\kappa = 0$	$\kappa = 10$	$\kappa = 20$	$\kappa = 30$
$\partial p_F/\partial \eta$	.13	15.	21.	15.	.31	-.95	-1.3	15.	22.	23.	15.
$\partial p_F/\partial \mu$	-.11	-40.	-66.	-40.	-.28	.56	.51	-40.	-61.	-61.	-37.
$\partial p_F/\partial \kappa$	.002	.33	.46	.33	.005	-.014	-.017	.33	.51	.61	.54

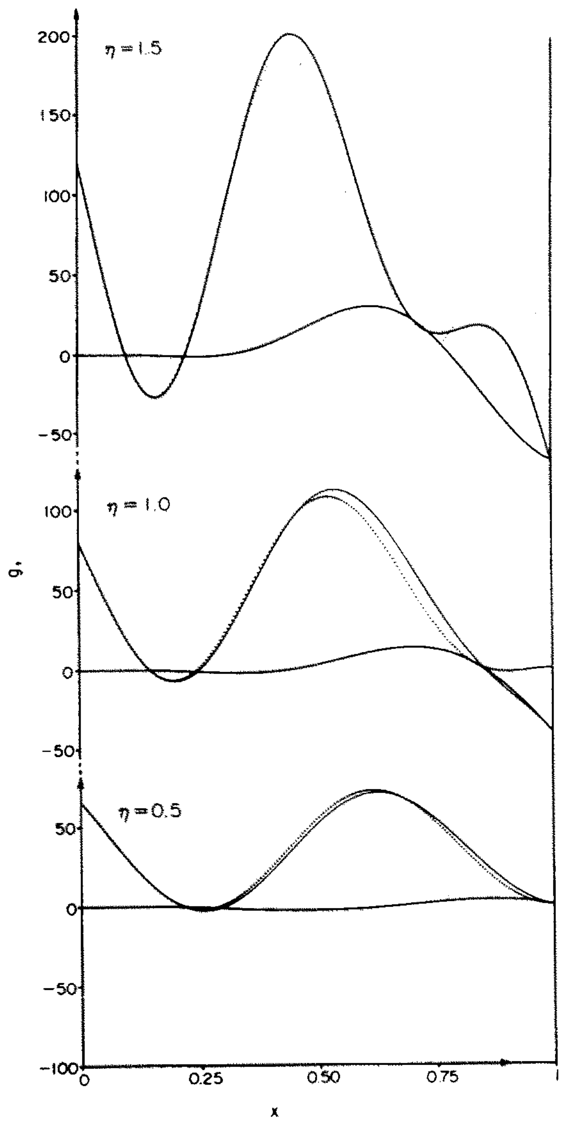

Fig. 2. Gradient function (4.17) for Beck's column *with partial follower force* ($\eta = 0.5, 1.0$ and 1.5). No end-mass and end-spring ($\mu = \kappa = 0$). Uniform beam ($m \equiv 1, s \equiv m^2$). Dotted curves give $2mv_F''u_F''$, curves starting at (0, 0) give ($-\omega_F^2 v_F u_F$), and the total gradient functions are also shown.

For a cantilever beam of nonuniform mass $m(x)$, bending stiffness $s(x)$, Kelvin–Voigt internal damping with coefficient γ, external viscous damping with coefficient β, loaded according to the distribution $pq(x)$, and with eigenvalue λ, the separated differential equation in space is

$$((su'')'' + pQu'') + \lambda(\beta u + \gamma(su'')'') + \lambda^2 mu = 0$$
$$\left(Q = \int_x^1 q(\xi)\,\mathrm{d}\xi\right) \tag{5.1}$$

with the boundary conditions

$$u(0) = u'(0) = 0$$
$$(1+\lambda\gamma)(su'')_{x=1} = (1+\lambda\gamma)(su'')'_{x=1} = 0 \tag{5.2}$$

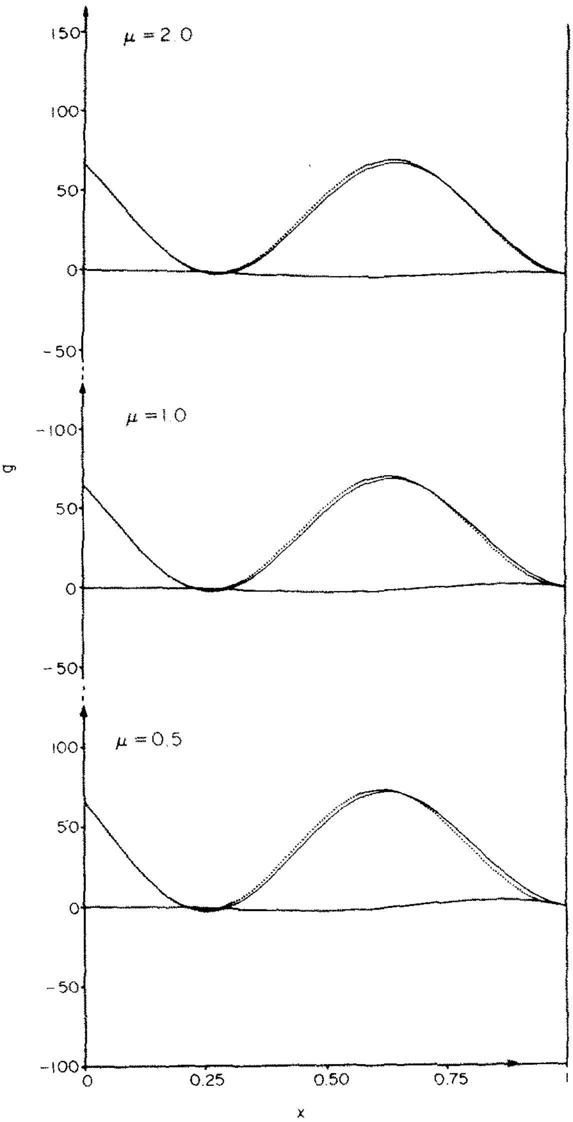

Fig. 3. Gradient function (4.17) for Beck's column *with end-mass* ($\mu = 0.5$, 1.0 and 2.0). No end-spring ($\kappa = 0$) and tangential force ($\eta = 1$). Uniform beam ($m \equiv 1, s = m^2$). Dotted curves give $2mv_F''u_F''$, curves starting at (0, 0) give $(-\omega_F^2 v_F u_F)$, and the total gradient functions are also shown.

Specific cases of uniform columns without damping ($\beta = \gamma = 0$) are known from the literature, i.e.

$$\text{Leipholz[22]: } q(\xi) = 1 \Rightarrow Q = 1 - x$$
$$\text{Hauger[23]: } q(\xi) = 1 - \xi \Rightarrow Q = \frac{1}{2}(1-x)^2 \tag{5.3}$$

and an optimization paper by Claudon[14] treats the Hauger column with damping.

The adjoint problem to (5.1)–(5.2) is described by the differential equation

$$((sv'')'' + p(Qv)'') + \lambda(\beta v + \gamma(sv'')'') + \lambda^2 mv = 0$$
$$\left(Q = \int_x^1 q(\xi)\,\mathrm{d}\xi\right) \tag{5.4}$$

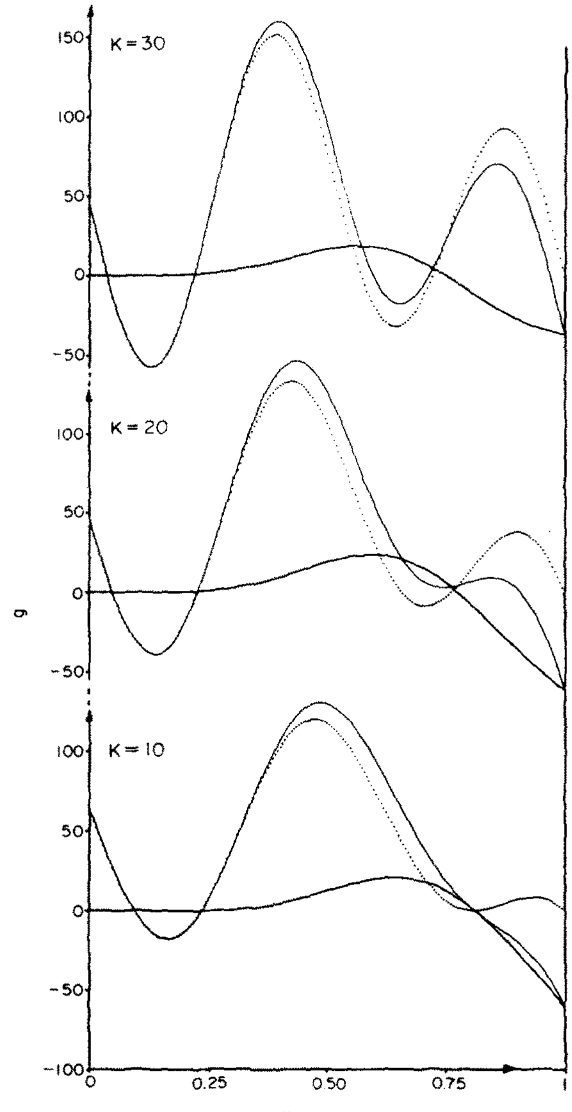

Fig. 4. Gradient function (4.17) for Beck's column *with end-spring* ($\kappa = 10, 20$ and 30). No end-mass ($\mu = 0$) and tangential force ($\eta = 1$). Uniform beam ($m \equiv 1, s = m^2$). Dotted curves give $2mv_F''u_F''$, curves starting at $(0, 0)$ give $(-\omega_F^2 v_F u_F)$, and the total gradient functions are also shown.

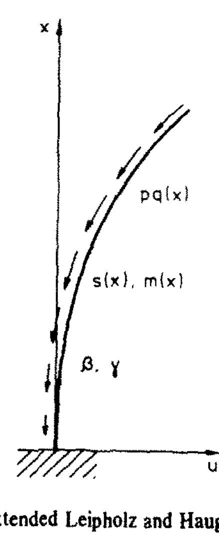

Fig. 5. Extended Leipholz and Hauger column.

with the boundary conditions

$$v(0) = v'(0) = 0$$
$$(1 + \lambda\gamma)(sv'')_{x=1} = ((1 + \lambda\gamma)(sv'')' - pqv)_{x=1} = 0 \tag{5.5}$$

For the Hauger column $q(1) = 0$, and (5.4)–(5.5) agree with [14].

Analysis for eigenvalues, eigenmodes and adjoint eigenmodes

Approximate solutions are obtained by expansions of the eigenmodes and adjoint eigenmodes, i.e. discretization into the linear combination factors of the expansions

$$u \approx \sum_{j=1}^{J} u_j\phi_j,$$
$$v \approx \sum_{i=1}^{J} v_i\psi_i. \tag{5.6}$$

Primarily we shall assume that the expansion functions u_j, v_i satisfy the kinematic boundary conditions only

$$u_j(0) = u_j'(0) = 0 \quad \text{for} \quad j = 1, 2, \ldots, J,$$
$$v_i(0) = v_i'(0) = 0 \quad \text{for} \quad i = 1, 2, \ldots, J. \tag{5.7}$$

Let's consider the functional $I(u, v)$ which is obtained from the scalar product (Lu, v) with the use of integration by parts

$$I(u, v) = \int_0^1 ((1 + \lambda\gamma)sv''u'' + pQvu'' + \lambda\beta vu + \lambda^2 mvu)\,\mathrm{d}x. \tag{5.8}$$

It is easy to see that stationarity of this functional with respect to arbitrary smooth variations δu, δv, satisfying kinematic boundary conditions (5.7), is equivalent to the boundary value problems (5.1)–(5.2), (5.4)–(5.5).

Using expansions (5.6) and taking variations in the form

$$\delta u = \delta\phi_j u_j, \quad \delta v = \delta\psi_i v_i, \quad i, j = 1, 2, \ldots J$$

we get

$$[L]\{\Phi\} = \{0\} \quad \text{and} \quad \{\Psi\}^T[L] = \{0\}^T$$
$$[L] = [S] + p[Q] + \lambda(\gamma[S] + \beta[C]) + \lambda^2[M] \tag{5.9}$$

where the elements of the involved matrices are defined by

$$\text{elements of the stiffness matrix } [S]\text{:} \quad s_{ij} = \int_0^1 sv_i''u_j''\,\mathrm{d}x \tag{5.10}$$

$$\text{elements of the load matrix } [Q]\text{:} \quad g_{ij} = \int_0^1 Qv_iu_j''\,\mathrm{d}x \tag{5.11}$$

$$\text{elements of the damping matrix } [C]\text{:} \quad c_{ij} = \int_0^1 v_iu_j\,\mathrm{d}x \tag{5.12}$$

$$\text{elements of the mass matrix } [M]\text{:} \quad m_{ij} = \int_0^1 mv_iu_j\,\mathrm{d}x \tag{5.13}$$

The approximate eigenvalue is then obtained by the determinant condition

$$|[L]| = |[L]^T| = 0, \tag{5.14}$$

and the corresponding eigenvectors and adjoint eigenvectors are then evaluated from eqn (5.9). This method we may call a Ritz method.

If u_j as well as v_i satisfies all the boundary conditions the method is the one proposed by Prasad and Herrmann[5], and if, furthermore, $v_i = u_i$ we have the classical Bubnov–Galerkin method. For the present solutions we choose as expansion functions the eigenfunctions which resemble those for the uniform cantilever beam

$$\begin{gathered} v_i = u_i = \cosh(a_i x) - \cos(a_i x) - f_i(\sinh(a_i x) - \sin(a_i x)), \\ f_i = (\cosh(a_i) + \cos(a_i))/(\sinh(a_i) + \sin(a_i)), \\ a_1 = 1.8751040687,\ a_2 = 4.6940911329,\ a_3 = 7.8547574382, \\ a_4 = 10.995407348,\ a_5 = 14.1371683910,\ a_6 = 17.2787595320. \end{gathered} \tag{5.15}$$

The same approach, but limited to two functions, was used by Hauger[23]. In Fig. 6 we show the resulting characteristic curves $\omega = \omega(p)$ and $\alpha = \alpha(p)$ obtained by two, four and six functions. We note, that the two mode expansion gives an appreciable error ($p_F = 158.2$), while the difference between the results of the four and the six mode expansions is rather small. These results confirm the flutter load $p_F = 150.6$ obtained by Claudon[13].

A number of different examples have been solved to obtain the information necessary for the sensitivity analysis to follow. In Figs. 7–9 we present the characteristic curves corresponding to extended versions of the Beck problem, the Leipholz problem, and the Hauger problem. We include external damping β and internal damping γ, and show solutions for non-uniform columns. The information related to the second eigenmode is omitted, being of minor interest, but as in Fig. 6 we show the solutions also in the domain of instability, which gives much more insight. The quantitative results for these problems are given in Table 3.

It is interesting to note how similarly the three different problems react to the introduction of damping. The relations between the uniform column and the column with linearly varying mass ($m = 1 - 0.9x$, $s = m^2$) are also very similar. Thus, the main difference between the results for the Beck, the Leipholz and the Hauger columns is the scale of the abscissa. The influence of internal damping, so nicely discussed by Bolotin and Zhinzher[24], is here illustrated in a number of different problems. At the flutter point ($\alpha_F = 0$) the curves $\alpha = \alpha(p)$ cross without changing the sign of the curvature, i.e. we have $\partial^2\alpha/\partial p^2 > 0$. This also holds for $\gamma \to 0$.

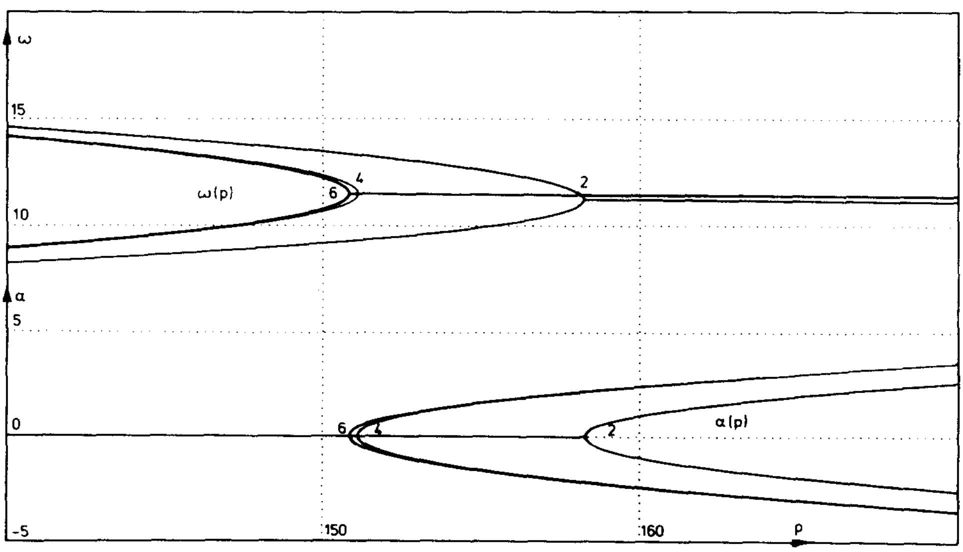

Fig. 6. Characteristic curves for the Hauger column obtained by two, four and six expansion functions.

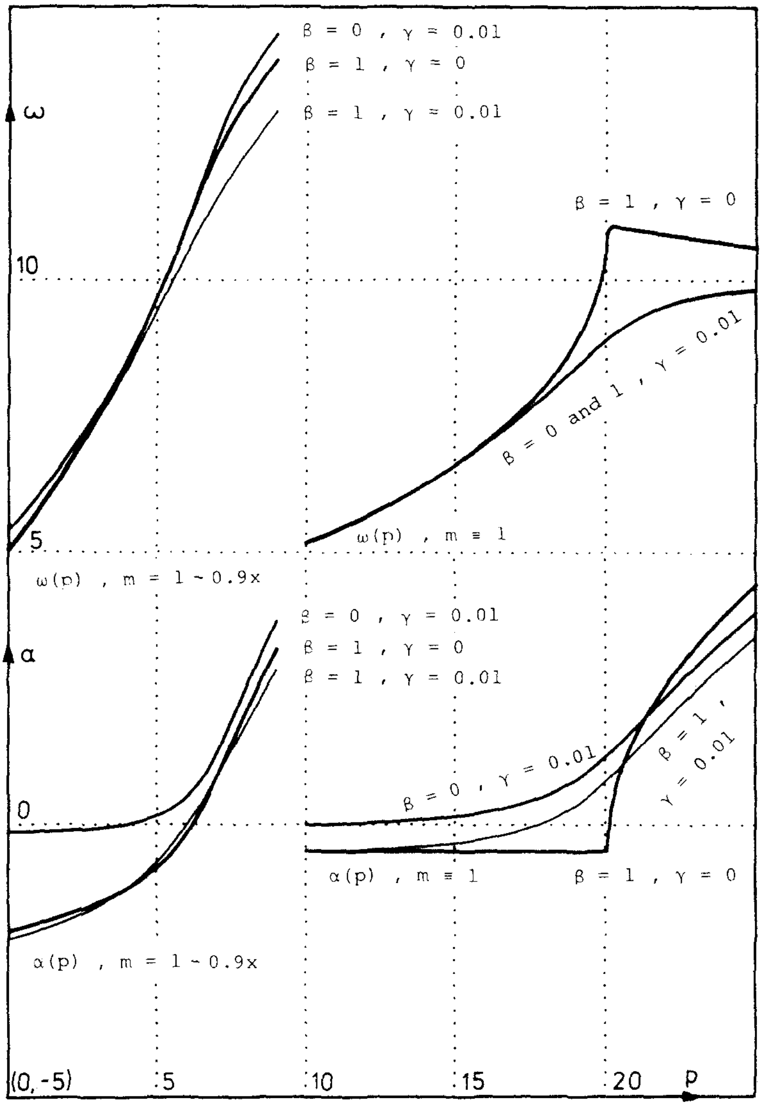

Fig. 7. Characteristic curves for the *Beck column* with external and internal damping, curves to the right for a uniform column, and curves to the left for a linearly tapered column.

Sensitivity analysis at the discretized level

As we have seen from the results an expansion in six (or even four) functions gives rather accurate results. For this case the sensitivities by (2.11)–(2.15) only involve matrix calculation of order six or four.

The necessary derivatives of the system matrix $[L]$, we get from the definitions (5.9)–(5.13),

$$\begin{aligned}
[L]_{,\lambda} &= \gamma[S] + \beta[C] + 2\lambda[M] \\
[L]_{,p} &= [Q] \\
[L]_{,t} &= (1+\lambda\gamma)[S]_{,t} + \lambda^2[M]_{,t} \\
&= (1+\lambda\gamma)\int_0^1 s_{,t}v_i''u_j''\,\mathrm{d}x + \lambda^2\int_0^1 m_{,t}v_iu_j\,\mathrm{d}x
\end{aligned} \tag{5.16}$$

and for the specific case of $\underline{s = m^2}$, the last equation is

$$[L]_{,t} = \int_0^1 ((1+\lambda\gamma)2mv_i''u_j'' + \lambda^2 v_iu_j)m_{,t}\,\mathrm{d}x. \tag{5.17}$$

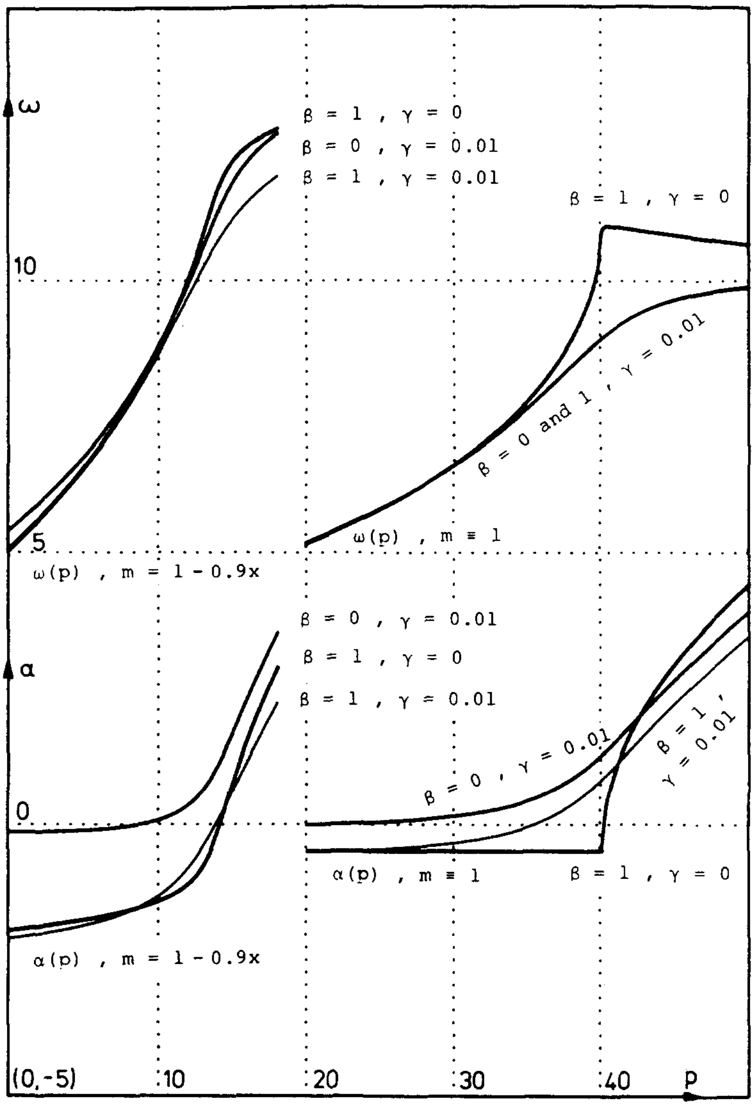

Fig. 8. Characteristic curves for the *Leipholz column* with external and internal damping, curves to the right for a uniform column, and curves to the left for a linearly tapered column.

For this case it follows from (2.11) that defining the gradient function $g = g(x)$ by

$$g(x) = -Re\left(\frac{\sum_i \sum_j ((1+\lambda\gamma)2mv_i''u_j'' + \lambda^2 v_i u_j)\psi_i\phi_j}{\{\Psi\}^T(\gamma[S] + \beta[C] + 2\lambda[M])\{\Phi\}}\right)$$
$$Re\left(\frac{\{\Psi\}^T[Q]\{\Phi\}}{\{\Psi\}^T(\gamma[S] + \beta[C] + 2\lambda[M])\{\Phi\}}\right) \tag{5.18}$$

we get

$$\delta P_{cr} = \int_0^1 g\delta m \, \mathrm{d}x, \tag{5.19}$$

analogously to (4.18).

In Figs. 10–12 we show the gradient function (5.18) corresponding to the results given in Table 3. Note the agreement of Fig. 10 with Fig. 2 for the very special case, which can be solved analytically, i.e. the uniform Beck column.

Table 3. Flutter loads and flutter frequencies obtained with four expansion functions

LOAD	$Q(x) \equiv 1$ (BECK column)				$Q(x) = 1 - x$ (LEIPHOLZ column)				$Q(x) = \frac{1}{2}(1-x)^2$ (HAUGER column)			
DAMPING	$\beta=0$ $\gamma=0$	$\beta=1$ $\gamma=0$	$\beta=0$ $\gamma=.01$	$\beta=1$ $\gamma=.01$	$\beta=0$ $\gamma=0$	$\beta=1$ $\gamma=0$	$\beta=0$ $\gamma=.01$	$\beta=1$ $\gamma=.01$	$\beta=0$ $\gamma=0$	$\beta=1$ $\gamma=0$	$\beta=0$ $\gamma=.01$	$\beta=1$ $\gamma=.01$
Uniform beam with $s = m^2$ and $m \equiv 1$												
Flutter load p_F	20.0	20.1	11.0	17.8	40.1	40.2	21.9	35.6	151.1	151.5	79.2	132.0
Flutter freq. ω_F	10.9	11.0	5.4	7.8	11.0	11.0	5.4	7.8	11.5	11.5	5.4	8.0
Linear tapered beam with $s = m^2$ and $m = 1 - 0.9x$												
Flutter load p_F	6.3	6.2	4.1	6.0	14.1	14.1	8.9	13.8	59.3	61.1	37.5	60.8
Flutter freq. ω_F	11.8	11.3	8.6	10.5	13.3	11.8	8.3	10.7	13.3	12.8	8.3	11.3

Sensitivities with respect to load distribution

In this section we have been dealing with different specific load distributions $q = q(x)$. An interesting question here concerns the sensitivity of the critical load to changes in the distributions. The variation δq gives rise to the increments δQ, $\delta\lambda$, δp, and from (5.1) we get

$$\delta p \int_0^1 vQu''\,dx + p\int_0^1 v\delta Qu''\,dx + \delta\lambda\int_0^1 (v\beta u + v\gamma(su'')'')\,dx + 2\lambda\delta\lambda\int_0^1 vmu\,dx = 0, \tag{5.20}$$

because the stationarity makes it possible to neglect the increments δv and δu.

At the critical load $p = p_{cr}$ we have $Re(\delta\lambda) = 0$ and $\lambda = i\omega_{cr}$, whereby (5.20) becomes

$$\delta p_{cr} Re\left(\frac{\int_0^1 vQu''\,dx}{z}\right) = -p_{cr} Re\left(\frac{\int_0^1 v\delta Qu''\,dx}{z}\right) \tag{5.21}$$

with z defined by

$$z = \int_0^1 ((\beta + i2\omega_{cr}m)vu + \gamma v(su'')'')\,dx. \tag{5.22}$$

From $Q = \int_x^1 q(\xi)\,d\xi$ we have $\delta Q = \int_x^1 \delta q\,d\xi$, and thus we may rewrite (5.21) as

$$\delta p_{cr} = \int_0^1 g\delta q\,dx, \tag{5.23}$$

where the gradient function $g = g(x)$ is defined by

$$g(x) = -p_{cr} Re\left(\frac{\int_0^x vu''\,d\xi}{z}\right)\Big/ Re\left(\frac{\int_0^1 vQu''\,dx}{z}\right). \tag{5.24}$$

Analogously, we obtain results for $\delta\omega_{cr}$, and the case without damping again gives simplified results.

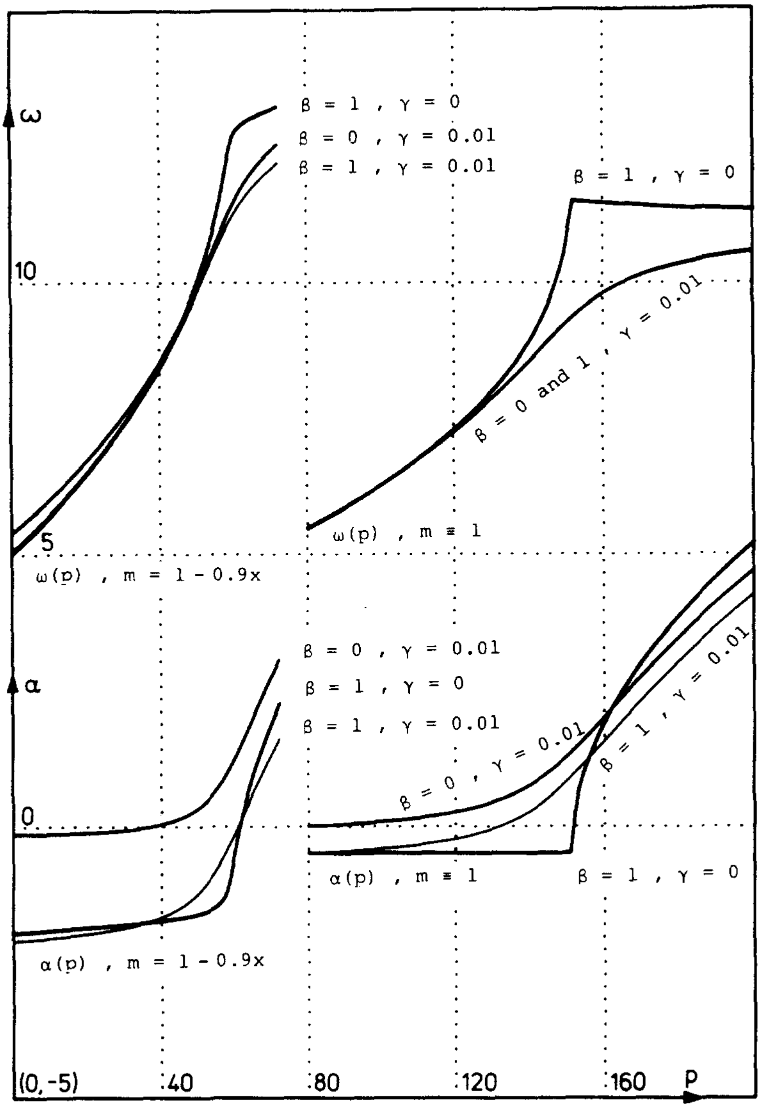

Fig. 9. Characteristic curves for the *Hauger column* with external and internal damping, curves to the right for a uniform column, and curves to the left for a linearly tapered column.

In matrix notation the result may be given analogously to (5.18) because, corresponding to (5.17), we have from (5.9)–(5.11)

$$[L]_{,t} = p\int_0^1 vu''Q_{,t}\,\mathrm{d}x. \tag{5.25}$$

6. CONCLUSION

Sensitivity analysis of mechanical systems is a subject of increasing importance. Besides being essential for solution of problems of optimal design, this analysis gives answers to many interesting questions.

For non-conservative problems of dynamic stability many earlier papers have given confusing results. Hopefully, the present paper may give some clarification. The sensitivity analyses performed in detail here illustrate the important information that can be obtained. The sensitivity of critical values of stability with respect to vanishing damping should be mentioned as an example demanding special analysis [24].

Finally, we would like once more to stress the point, that a sensitivity analysis demands only comparatively few calculations because all the necessary data are available from the ordinary

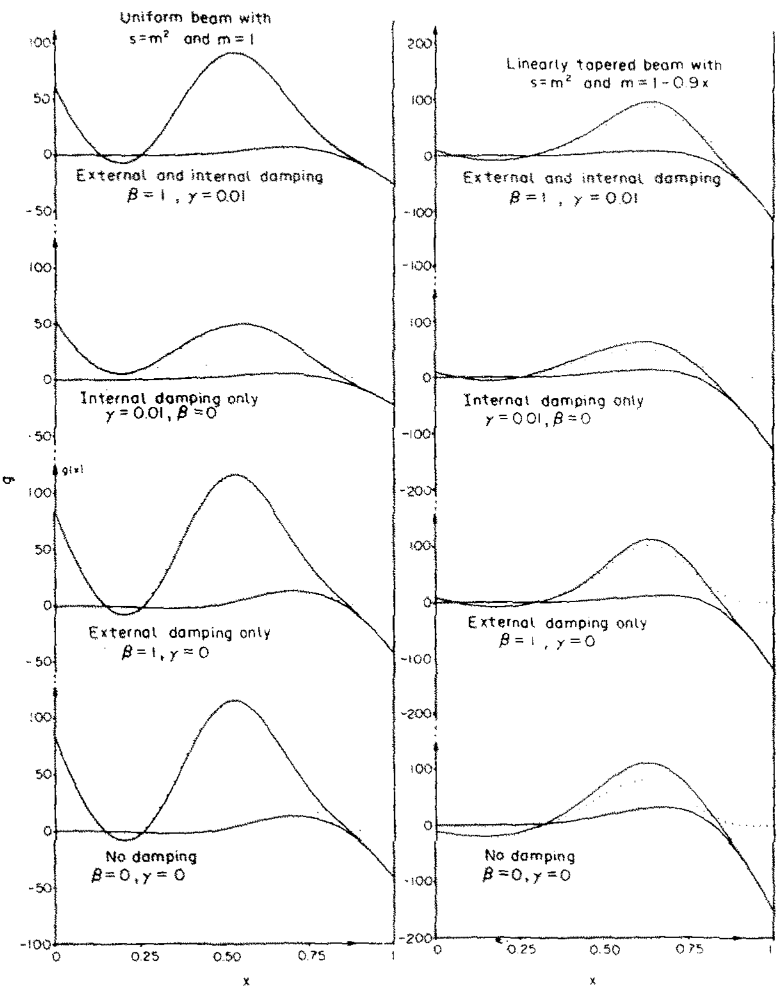

Fig. 10. Gradient function (5.18) for the *Beck column* with external and internal damping. Note the different scales for the uniform column and the linearly tapered column. Gradient function for non-structural masses are also given separately as in Fig. 2–4. The lower, left case is identical to the case in the middle of Fig. 2.

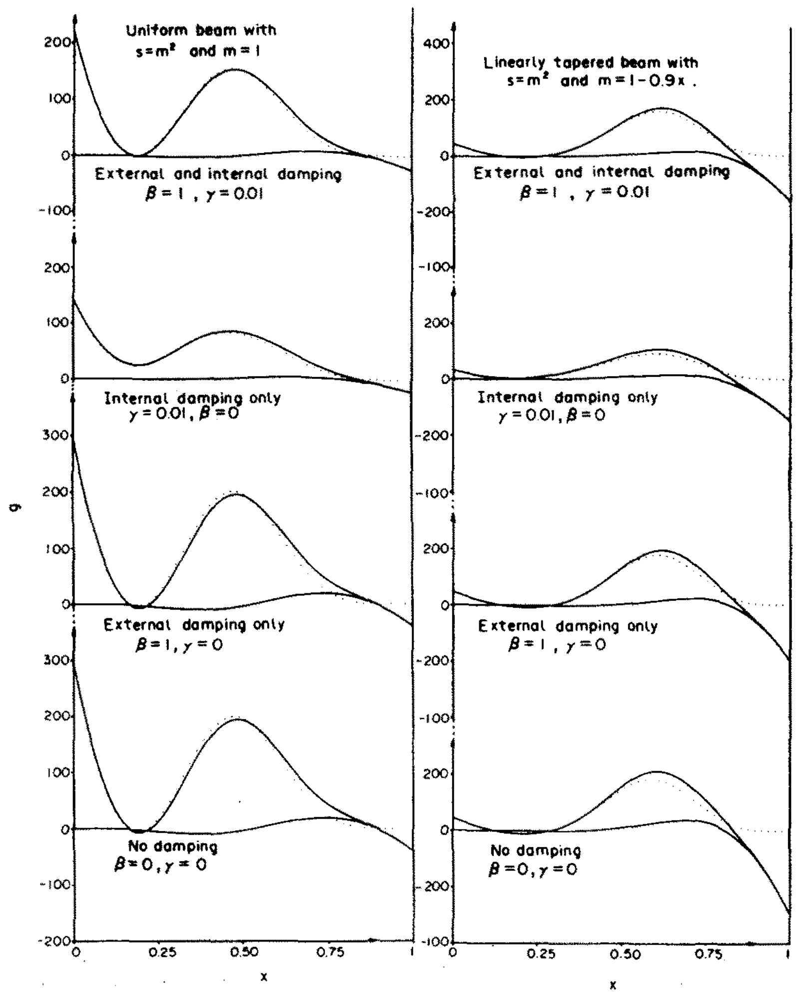

Fig. 11. Gradient function (5.18) for the *Leipholz column* with external and internal damping. Note the different scales for the uniform column and the linearly tapered column.

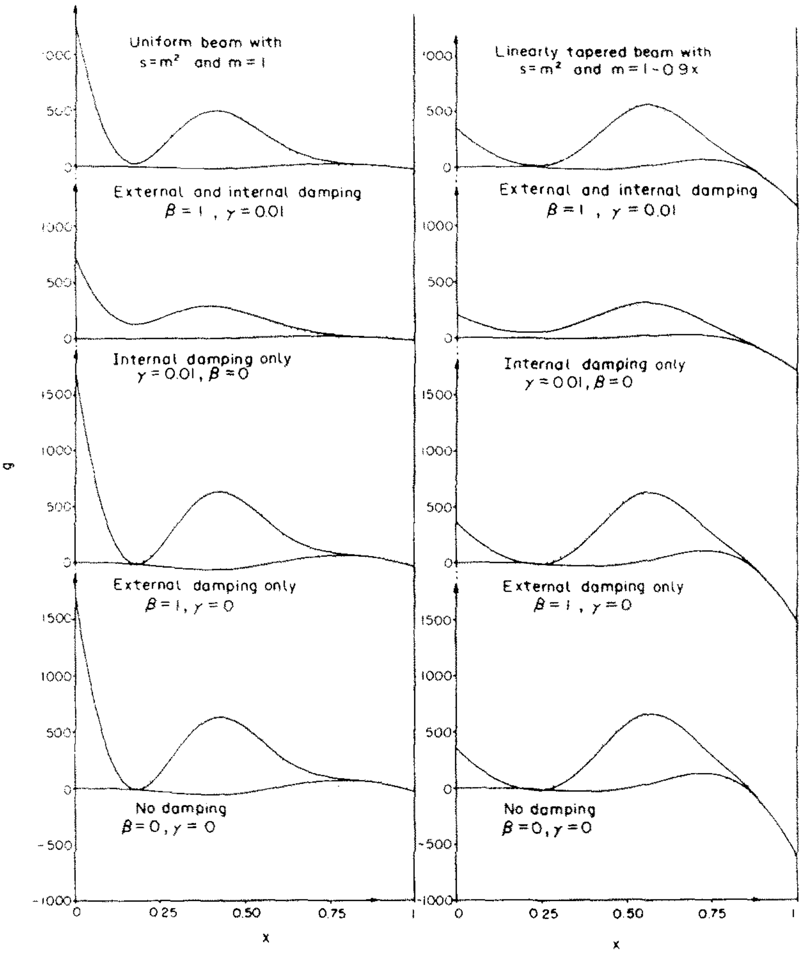

Fig. 12. Gradient function (5.18) for uniform and linearly tapered *Hauger column* with external and internal damping.

analysis. For non-conservative problems this data, however, must include the solution to the adjoint problem.

The intension of the paper has been to focus on sensitivity analysis unrelated to optimal design, but a natural extension of the present work is then to use the results in optaining optimal designs.

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REFERENCES

1. V. V. Bolotin, *Nonconservative Problems of the Theory of Elastic Stability*, p. 324, Pergamon Press, Oxford (1963).
2. H. Ziegler, *Principles of Structural Stability*, p. 150, Blaisdell, Waltham, Mass. (1968).
3. H. Leipholz: *Stability of Elastic Systems*, Sijthoff & Noordhoof, Amsterdam (1980).
4. H. Leipholz, Analysis of nonconservative, nonholonomic system. *Proc. IUTAM Cong.*, Toronto, Canada, Springer-Verlag, New York (1980).
5. S. N. Prasad and G. Herrmann, The usefulness of adjoint systems in solving non-conservative stability problems of elastic continua. *Int. J. Solids Structures* **5**, 727–735 (1969).
6. S. N. Prasad and G. Herrmann, Adjoint variational methods in non-conservative stability problems. *Int. J. Solids Structures* **8**, 29–40 (1972).
7. E. J. Haug and J. Cea, Optimization of distributed parameter structures. *Proc. NATO Advanced Study Inst.*, 2 Vols. Sijthoff & Noordhoof, Amsterdam (1981).
8. H. Ashley and S. C. McIntosh, Application of aeroelastic constraints in structural optimization. *Proc. IUTAM Cong.*, Stanford, Springer-Verlag, New York (1968).
9. S. C. McIntosh and H. Ashley, On the optimization of discrete structures with aeroelastic constraints. *Comput. & Structures* **8**, 411–419 (1978).
10. K. Vepa, Generalization of an energetic optimality condition for non-conservative systems. *J. Struct. Mech.* **2**, 229–257 (1973).
11. R. H. Plaut, Optimal design for stability under dissipative, gyroscopic, or circulatory loads, *Proc. IUTAM Symp.*, Warsaw, Poland, (Edited by Sawczuk & Mroz) Springer-Verlag, 168–180 (1975).

12. T. A. Weisshaar and R. H. Plaut, Structural optimization under non-conservative loading. *Proc. NATO Advanced Study Inst.*, pp. 843–864. Sijthoff & Noordhoff, Amsterdam (1981).
13. J. L. Claudon, Characteristic curves and optimum design of two structures subjected to circulatory loads. *J. Mécanique* **14**, 531–543 (1975).
14. J. L. Claudon, Détermination et maximisation de la charge critique d'une colonne de Hauger en Présence d'amortissement. *ZAMP* **29**, 226–236 (1978).
15. J. L. Claudon and M. Sunakawa, Optimizing distributed structures for maximum flutter load. *AIAA J.* **19**, 957–959 (1981).
16. F. Odeh and I. Tadjbakhsh, The shape of the strongest column with a follower load. *J. Opt. Theory & Appl.* **15**, 103–118 (1975).
17. H. Hanaoka and K. Washizu, Optimum design of Beck's column. *Comput. & Structures* **11**, 473–480 (1980).
18. A. P. Seyranian, Sensitivity analysis and optimization of aeroelastic stability characteristics (in Russian). Inst. for Prob. in Mech., USSR Academy of Sci, No. 162, Moscow, 1980, *Int. J. Solids & Structures* **18**, 791–807 (1982).
19. P. Pedersen, Design with several eigenvalue constraints by finite element and linear programming, *J. Struct. Mech.* **10**, 243–271 (1982).
20. V. G. Bun'kov, Calculation of optimal flutter characteristics by gradient method (in Russian). *Proc. of CAHI* **1166**, 3–43 (1969).
21. P. Pedersen, Influence of boundary conditions on the stability of a column under non-conservative load. *Int. J. Solids Structures* **13**, 445–455 (1977).
22. H. Leipholz, Die Knicklast des einseitig eingespannter Stabes mit gleichmässig verteilter, tangentialer Längsbelastung. *ZAMP* **13**, 581–589 (1962).
23. W. Hauger, Die Knicklasten Elastischer Stäbe unter Gleichmässig Verteilten und Linear Veränderlichen, Tangentialen Druckkräften. *Ingenieur-Archiv* **35**, 221–229 (1966).
24. V. V. Bolotin and N. I. Zhinzher, Effects of damping on stability of elastic systems subjected to nonconservative forces. *Int. J. Solids & Structures* **5**, 965–989 (1969).